The role of symmetry for finding black holes in scalar-tensor-theories

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Plan of the talk

1. Scalar Tensor Theories (STT)
2. Historical overview of solutions for which symmetry has proven to be of great help.
3. Model with a pure geometric constraint but without a conformal scalar field action (construction and solutions).
4. Model with a simple scalar field equation (construction and solutions).
5. Conclusions and Further prospects.

Scalar Tensor Theories (STT)

1. Scalar tensor theories are one of the simplest modified gravity theories which extend GR with one (or more) scalar degrees of freedom.
2. Horndeski theory : The most general (single) scalar-tensor theory with second order equations of motion $\Longrightarrow$ absence of Ostrogradski ghosts [G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974)]. The action is given by $\int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i$ where

$$\mathcal{L}_2 = K(\phi, X), \qquad \mathcal{L}_3 = -G_3(\phi, X)\Box\phi,$$
$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$$
$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5,X}}{6}\Big[(\Box\phi)^3$$
$$-3\left(\Box\phi\right)\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\Big]$$

where $X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ and $G_{i,X} = \frac{dG_i}{dX}$.

Black hole solutions

Goal : Find static spherically symmetric black hole solutions within an Ansatz of the form

$$ds^2 = -N(r)^2 F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right), \quad \phi = \phi(r)$$

or in isotropic coordinates

$$ds^2 = -H(r)dt^2 + G(r)\left[dr^2 + r^2\left(d\theta^2 + \sin(\theta)^2 d\varphi^2\right)\right], \quad \phi = \phi(r)$$

$\rightarrow$ Nonlinearities make difficult to find exact analytic solutions.

Historical overview of solutions

- Einstein gravity coupled to a massless scalar field

$$S=\int d^4x\sqrt{-g}\left(R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right)\qquad G_4=1,\quad K=X.$$

$\rightarrow$ Most general static solution in isotropic coordinates [B. Xanthopoulos and T. Zannias, , Phys. Rev. D **40**, 2564 (1989).] $\longrightarrow$ Naked singularity (unless the scalar field vanishes) ; this result is covered by the no-hair theorem [J. E. Chase, Commun. Math. Phys. 19, 276 (1970)].

$\rightarrow$ The clue of the derivation $\Box\phi=0\longrightarrow$ first integral.

Historical overview of solutions

- Scalar field nonminimally coupled

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\xi}{2}R\phi^2 - V(\phi)\right)$$

- Belongs to Horndeski theory $K = X - V(\phi)$ and $G_4 = 1 - \frac{\xi}{2}\phi^2$.

$\rightarrow$ The parameter ξ measures the strength of the nonminimal gravitational coupling.

$\rightarrow$ Minimal case $\xi = 0$ and $V' \geq 0$, no scalar-hair theorem [J. D. Bekenstein, Phys. Rev. Lett **28**, 452 (1972).]

$\rightarrow$ No scalar hair theorem for definite positive potential and for $\xi < 0$ and $\xi \geq \frac{1}{2}$ [A. E. Mayo and J. D. Bekenstein,, Phys. Rev. D **54**, 5059 (1996).]

Historical overview of solutions

$\rightarrow$ For the conformal coupling $\xi = \frac{1}{6}$ and $V = 0$, Bocharova, Bronnikov and Melnikov found a black hole solution [N. Bocharova, K. Bronnikov and V. Melnikov, Vest. Moks. Univ. Fiz. Astron. **6**, 706 (1970).] and [J. D. Bekenstein, Ann. Phys **82**, 535 (1974).]

- The BBMB solution is

$$ds^2 = -(1 - \frac{M}{r})^2 dt^2 + \frac{dr^2}{(1 - \frac{M}{r})^2} + r^2 d\Omega_2^2,$$

$$\phi(r) = \pm \frac{M}{r - M}.$$

$\rightarrow$ The metric is like the extremal Reissner-Nordstrom

$\rightarrow$ Scalar field blows up at the horizon $r_h = M$.

$\rightarrow$ Uniqueness of the BBMB solution [B. C. Xanthopoulos and T. Zannias, J. Math. Phys **32**, 1875 (1991).]

$\rightarrow$ The BBMB has a (very scanty) hair from the dichotomic parameter $\pm$ (due to the discrete symmetry $\phi \rightarrow -\phi$)

Historical overview of solutions

• The clue of the derivation and of the uniqueness theorem of the BBMB solution is the conformal invariance of the matter source action

$$S_M = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}R\phi^2\right)$$

$\to$ Conformal transformations : $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$ and $\phi \to e^{-\sigma}\phi$
$\Longrightarrow S_M \to S_M + \text{b.t.}$

$\to$ From the conformal invariance, the trace of the matter stress tensor vanishes $T_\mu^\mu = 0$, and from the Einstein equations $G_{\mu\nu} = T_{\mu\nu} \Longrightarrow R = 0$ (pure geometric constraint) and $\Box\phi = \frac{1}{6}R\phi \Longrightarrow \Box\phi = 0$ (first integral).

$\to$ This permits the derivation of the most general static spherically symmetric asymptotically flat solution (BBMB) [B. C. Xanthopoulos and T. Zannias, J. Math. Phys **32**, 1875 (1991).].

Historical overview of solutions

• Einstein conformal scalar field equations in arbitrary dimension D

$$S_M=\int d^Dx\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{\xi_D}{2}R\phi^2\right),\quad \xi_D=\frac{(D-2)}{4(D-1)}$$

$\rightarrow$ Conformal transformations : $g_{\mu\nu}\rightarrow e^{2\sigma}g_{\mu\nu}$ and $\phi\rightarrow e^{\frac{2-D}{2}\sigma}\phi$
$\Longrightarrow S_M\rightarrow S_M+\text{b.t.}$
$\rightarrow$ From the conformal invariance, and from the Einstein equations $G_{\mu\nu}=T_{\mu\nu}\Longrightarrow R=0$ (pure geometric constraint) and $\Box\phi=\xi_D R\phi\Longrightarrow\Box\phi=0$ (first integral).
$\rightarrow$ This permits the derivation of the most general static spherically symmetric asymptotically flat solution [C. Klimcik, J. Math. Phys **34**, 5 (1993).]. Black hole only in $D=4$ (BBMB).

Historical overview of solutions

- Self-interacting conformal scalar field

$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}R\phi^2 - \alpha\phi^4\right)$$

→ Conformal potential $V \propto \phi^4$.
→ A black hole solution with $\Lambda > 0$ exists [C. Martinez, R. Troncoso and J. Zanelli, Phys. Rev. D **67**, 024008 (2003).]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = -\frac{\Lambda}{3}r^2 + (1 - \frac{M}{r})^2$$

$$\phi(r) = \frac{M}{r - M}$$

provided that $\alpha = -\Lambda/72$.

Historical overview of solutions

$\rightarrow$ From the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \Longrightarrow R = 4\Lambda$ (pure geometric constraint) but
$\Box\phi = \frac{1}{6}R\phi + 4\alpha\phi^3 \Longrightarrow \Box\phi \neq 0$ (no more a first integral).
$\rightarrow$ The uniqueness of the solution is an open problem.

$\rightarrow$ From these different examples, one can appreciate that the solutions can be found analytically in the case where

1. Pure geometric constraint (due to the conformal invariance of the scalar field action) and which restricts the allowed possible spacetimes or/and
2. Scalar field equation "simple" to integrate.

• We will see a model where these two criteria hold but with a scalar field action that is not conformally invariant.

Model with a pure geometric constraint and a "simple" scalar field equation

- Let us generalize the standard conformal action

$$S = \int d^4x\sqrt{-g}\Big\{R - 2\Lambda - 6\beta\left(\frac{R}{6}\phi^2 + (\partial\phi)^2\right) - 2\lambda\phi^4$$
$$-\alpha\left[\ln(\phi)\mathcal{G} - \frac{4G^{\mu\nu}\phi_\mu\phi_\nu}{\phi^2} - \frac{4\Box\phi(\partial\phi)^2}{\phi^3} + \frac{2(\partial\phi)^4}{\phi^4}\right]\Big\}$$

where $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is the Gauss-Bonnet density. It belongs to Horndeski theory. Here the $\alpha-$contribution breaks the conf. invariance of the matter action but the scalar field equation is still conformally invariant.
$\rightarrow$ Look for a solution within the following ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad \phi = \phi(r).$$

1. Pure geometric constraint $R - 4\Lambda + \frac{\alpha}{2}\mathcal{G} = 0 \Longrightarrow$

$$f(r) = 1 + \frac{r^2}{2\alpha}\left[1 \pm \sqrt{1 + 4\alpha\left(\frac{2M}{r^3} - \frac{q}{r^4} + \frac{\Lambda}{3}\right)}\right],$$

where M (mass) and q (kind of charge) are two integration constants.

2. Scalar field equation "easy" to integrate (for $\alpha \neq 0$)

$$\left(\frac{\phi'}{\phi^2}\right)'\left(f\left[(r\phi)'\right]^2 - \phi^2\left(1 + \frac{\beta}{2\alpha}r^2\phi^2\right)\right) = 0$$

Two disconnected branches of solutions [P. G. S. Fernandes, Phys. Rev. D **103**, 104065, (2021).]

1. First branch :

$$\lambda=\frac{\beta^2}{4\alpha},\qquad q=-2\alpha,\qquad \phi(r)=\frac{\sqrt{-\frac{2\alpha}{\beta}}}{r}$$

2. Second branch : The scalar field equation

$$f\left[(r\phi)'\right]^2-\phi^2(1+\frac{\beta}{2\alpha}r^2\phi^2)=0$$

by means of the change $r\phi=Ah(\int\frac{dr}{r\sqrt{f(r)}})$ becomes a separable equation $\frac{dh}{h\sqrt{1+\frac{A^2\beta}{2\alpha}h^2}}=\pm dr$

$$\lambda=\frac{3\beta^2}{4\alpha},\qquad q=0,\qquad \phi(r)=\frac{\sqrt{-\frac{2\alpha}{\beta}}}{r\cosh(c\pm\int\frac{dr}{r\sqrt{f(r)}})}$$

where the constant c is a sort of hair (consequence of the conf. invariance of the scalar field equation).

Generalized or Non-Noetherian conformal scalar field

• The clue behind the existence of these solutions : Geometric constraint is due to the conformal symmetry of the scalar field equation without necessarily having a conformally invariant matter source action. $\longrightarrow$ Generalized [P. G. S. Fernandes, Phys. Rev. D **103**, 104065, (2021).] or Non-Noetherian conformal scalar field [E. Ayón-Beato and M. H, arXiv :2305.09806 [hep-th], (2023).]

$\rightarrow$ For convenience, let us work in the "exp. frame" $\phi \rightarrow e^{\phi}$.
$\rightarrow$ The conformal transformations become $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$ and $\phi \rightarrow \phi - \sigma$ or infinitesimally $\delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}$ and $\delta_\sigma \phi = -\sigma$. A SST with a conformally invariant scalar field equation

$$\delta_\sigma S = \int \left(\frac{\delta S}{\delta g_{\mu\nu}} \delta_\sigma g_{\mu\nu} + \frac{\delta S}{\delta \phi} \delta_\sigma \phi \right) = \int \underbrace{\left(2 g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} - \frac{\delta S}{\delta \phi} \right)}_{\phi-\text{independent}} \sigma$$

Generalized or Non-Noetherian conformal scalar field

$\rightarrow$ If the scalar field equation is conformally invariant $\Longrightarrow$

$$\boxed{-2g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}+\frac{\delta S}{\delta\phi}=\text{Pure Geometric Equation}}$$

$\rightarrow$ A scalar quantity $I(\phi, g)$ under a conformal transformation $g_{\mu\nu}\rightarrow e^{2\sigma}g_{\mu\nu}$ and $\phi\rightarrow\phi-\sigma$ becomes

$$I(\phi,g)\rightarrow I(\phi-\sigma, e^{2\sigma}g)$$

Conformal invariance of the scalar quantity $(\sigma=\phi)\Longrightarrow$

$$I(\phi,g)=I(0,\tilde{g}),\qquad \text{auxiliary metric } \tilde{g}=e^{2\phi}g$$

Conclusion : The only conf. inv. quantities of STT are purely geometric quantities built out of the auxiliary metric $(\delta_\sigma\tilde{g}=0)$.

In our case

$$I(0,\tilde{g}) = -8\lambda - 2\beta\tilde{R} - \alpha\tilde{\mathcal{G}}$$

and, by means of an homotopy calculation $\Longrightarrow$

$$S_M = \int d^4x\sqrt{-g}\Big\{-2\lambda\phi^4 - 6\beta\left(\frac{R}{6}\phi^2 + (\partial\phi)^2\right)$$
$$-\alpha\left[\ln(\phi)\mathcal{G} - \frac{4G^{\mu\nu}\phi_\mu\phi_\nu}{\phi^2} - \frac{4\Box\phi(\partial\phi)^2}{\phi^3} + \frac{2(\partial\phi)^4}{\phi^4}\right]\Big\}$$

- The real challenge is to determine scalar quantities built out of the auxiliary metric that come from an action principle. In [E. Ayón-Beato and M. H, arXiv :2305.09806 [hep-th], (2023).], we have determined the most general action in four dimensions that gives rise to a non-Noetherian conformal scalar field satisfying a second-order equation.

Model with a "simple" scalar field equation

A priori forget about the symmetry (conformal or shift) and ask a similar factorization for the scalar field equation for an ansatz of the form $ds^2 = -f(r)dt^2 + dr^2/f(r) + r^2 d\Omega_2^2$ and $\phi = \phi(r)$ [E. Babichev, C. Charmousis, M. H. and N. Lecoeur, [arXiv :2303.04126 [gr-qc]].]

$$\int \mathrm{d}^4x\sqrt{-g}\Big\{ (1+W(\phi))\,R - \frac{1}{2}V_k(\phi)(\nabla\phi)^2 + Z(\phi) + V(\phi)\,\mathcal{G}$$
$$+V_2(\phi)\,G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V_3(\phi)(\nabla\phi)^4 + V_4(\phi)\,\Box\phi\,(\nabla\phi)^2 \Big\}.$$

Previous case corresponds

$$W = -\beta\mathrm{e}^{2\phi}, \quad V_k = 12\beta\mathrm{e}^{2\phi}, \quad Z = -2\lambda\mathrm{e}^{4\phi} - 2\Lambda,$$
$$V = -\alpha\phi, \quad V_2 = 4\alpha = V_4, \quad V_3 = 2\alpha,$$

1. The combination $E_t^t - E_r^r = 0$ can be factorized as

$$\left[\frac{\phi''}{(\phi')^2} - 1\right]\left[r^2 W_\phi + 4\left(1-f\right)V_\phi + 2frV_2\phi' + fr^2 V_4\left(\phi'\right)^2\right] = 0$$

   provided that the potentials V_k and V_i can be parameterized in terms of the Einstein-Hilbert and Gauss-Bonnet potentials W and $V \Longrightarrow$ A priori a three-parametric (parameterized by W, V and Z) class of possible "integrable" theories.

2. One has to fix the potentials W, V and Z s. t. the two remaining equations admit the same metric function f (in the case of the first branch).

For the first branch, the two independent equations, $\mathcal{E}_{rr}=0$ and $\mathcal{E}_{\theta\theta}=0$ can be integrated once and twice respectively to give

$$\mathcal{E}_{rr}\propto I_1'(r)\,,\quad \mathcal{E}_{\theta\theta}\propto I_2''(r)\,,$$

with

$$I_1(r)=f^2\left(r^2V\right)'''-f\left(2r\left(1+\mathcal{W}'\right)+4V'+r^2\mathcal{W}''\right)+2r+2\mathcal{W}+r\mathcal{Z}'-\mathcal{Z},$$
$$I_2(r)=f^2\,(rV)''-fr\left(1+\mathcal{W}'\right)+\mathcal{Z},$$

where $W=\mathcal{W}'$ and $rZ=\mathcal{Z}''$.

The two quadratic equations defining f must be proportional with a proportional factor $2\mu(r)$.

Case of $\mu = 1$

For a proportionality factor $\mu = 1$

$$W = -\beta_4 e^{2\phi} - \beta_5 e^{3\phi}, \qquad Z = -2\Lambda - 2\lambda_4 e^{4\phi} - 2\lambda_5 e^{5\phi},$$
$$V = -\alpha_4 \phi - \alpha_5 e^{\phi},$$

and the resulting action is given by

$$S = \int d^4x\sqrt{-g}\Big\{R - 2\Lambda - 2\lambda_4 e^{4\phi} - 2\lambda_5 e^{5\phi} - \beta_4 e^{2\phi}\left(R + 6(\nabla\phi)^2\right)$$
$$-\beta_5 e^{3\phi}\left(R + 12(\nabla\phi)^2\right) - \alpha_4\left(\phi\mathcal{G} - 4G^{\mu\nu}\phi_\mu\phi_\nu - 4\Box\phi(\nabla\phi)^2 - 2(\nabla\phi)^4\right)$$
$$-\alpha_5 e^{\phi}\left(\mathcal{G} - 8G^{\mu\nu}\phi_\mu\phi_\nu - 12\Box\phi(\nabla\phi)^2 - 12(\nabla\phi)^4\right)\Big\},$$

The resulting action is a linear combination of the four-dimensional non Noetherian conformal action and a Lagrange density that defines a Noetherian conformal action in five dimensions.

Case of $\mu = 1$: Solutions of the first branch

Now, the theory being fixed let's use the Eddington Finkelstein coordinates (for latter convenience)

$$ds^2 = -f(r)\,du^2 - 2dudr + \frac{r^2 d\theta^2}{1-\kappa\theta^2} + r^2\theta^2 dy^2, \quad \phi(r) = \ln\left(\frac{\eta}{r}\right).$$

where $\kappa = \pm 1$ or $\kappa = 0$ and η is a constant.

Einstein equations

Evaluating the scalar field $\phi(r)$ in the Einstein equations E_{uu},

$$E_{uu} \propto \left(\frac{\alpha_4 f^2}{r} - \left(r + \frac{\beta_5\eta^3}{2r^2} + \frac{2\kappa\alpha_5\eta}{r^2} + \frac{2\alpha_4\kappa}{r}\right) f + \frac{1}{r^2}\left(\frac{\lambda_5\,\eta^5}{2} + \frac{\beta_5\,\eta^3\kappa}{2}\right) + \frac{1}{r}\left(\lambda_4\eta^4 + \beta_4\,\kappa\,\eta^2\right) - \frac{r^3\Lambda}{3} + \kappa r\right)'.$$

$$\Longrightarrow \frac{\alpha_4 f^2}{r} - \left(r + \frac{\beta_5\eta^3}{2r^2} + \frac{2\kappa\alpha_5\eta}{r^2} + \frac{2\alpha_4\kappa}{r}\right) f + \frac{1}{r^2}\left(\frac{\lambda_5\,\eta^5}{2} + \frac{\beta_5\,\eta^3\kappa}{2}\right) + \frac{1}{r}\left(\lambda_4\eta^4 + \beta_4\,\kappa\,\eta^2\right) - \frac{r^3\Lambda}{3} + \kappa r + C_1 = 0.$$

Evaluating the scalar field $\phi(r)$ in the Einstein equations $E_{\theta\theta}$,

$$E_{\theta\theta} \propto \left[\frac{\alpha_4}{r} f^2 - \left(r - \frac{\beta_5\eta^3}{r^2} - \frac{\beta_4\eta^2}{r}\right) f - \frac{\lambda_5\eta^5}{3r^2} - \frac{\lambda_4\eta^4}{r} - \frac{r^3\Lambda}{3}\right]''.$$

$$\Longrightarrow \frac{\alpha_4 f^2}{r} - \left(r - \frac{\beta_5\eta^3}{r^2} - \frac{\beta_4\eta^2}{r}\right) f - \frac{\lambda_5\eta^5}{3r^2} - \frac{\lambda_4\eta^4}{r} - \frac{r^3\Lambda}{3} + C_3\, r + C_2 = 0,$$

where C_1, C_2 and C_3 are constants of integration.

Compatibility and Black Hole Solution

$$\frac{\alpha_4 f^2}{r}-\left(r+\frac{\beta_5\eta^3}{2r^2}+\frac{2\kappa\alpha_5\eta}{r^2}+\frac{2\alpha_4\kappa}{r}\right)f+\frac{1}{r^2}\left(\frac{\lambda_5\,\eta^5}{2}+\frac{\beta_5\,\eta^3\kappa}{2}\right)$$
$$+\frac{1}{r}\left(\lambda_4\eta^4+\beta_4\,\kappa\,\eta^2\right)-\frac{r^3\Lambda}{3}+\kappa r+C_1=0,$$

$$\frac{\alpha_4 f^2}{r}-\left(r-\frac{\beta_5\eta^3}{r^2}-\frac{\beta_4\eta^2}{r}\right)f-\frac{\lambda_5\eta^5}{3r^2}-\frac{\lambda_4\eta^4}{r}-\frac{r^3\Lambda}{3}+C_3\,r+C_2=0,$$

Compatibility relations for coupling constants

$$\beta_5\eta^2=-\frac{4}{3}\alpha_5\kappa,\quad \beta_4\eta^2=-2\alpha_4\kappa,\quad \lambda_5\,\eta^2=-\frac{3}{5}\beta_5\kappa,\quad \lambda_4\eta^2=-\frac{1}{2}\beta_4\kappa$$
$$C_3=\kappa,\quad C_1=C_2=-2M.$$

where M is a constant of integration. We obtain the following metric function

$$f(r)=\kappa+\frac{2\alpha_5\eta\kappa}{3r\alpha_4}+\frac{r^2}{2\alpha_4}\left(1\pm\sqrt{\left(1+\frac{4\alpha_5\eta\kappa}{3r^3}\right)^2+4\alpha_4\left(\frac{\Lambda}{3}+\frac{2M}{r^3}+\frac{2\alpha_4\kappa^2}{r^4}+\frac{8\alpha_5\eta\kappa^2}{5r^5}\right)}\right),$$

Particular case $\kappa = 0$

For the case $\kappa = 0$, the theory is restricted by $\lambda_4 = \beta_4 = \lambda_5 = \beta_5 = 0$

$$S = \int d^4x\sqrt{-g}\Big\{R - 2\Lambda - \alpha_4\left(\phi\mathcal{G} - 4G^{\mu\nu}\phi_\mu\phi_\nu - 4\Box\phi(\nabla\phi)^2 - 2(\nabla\phi)^4\right)$$

$$-\alpha_5 e^{\phi}\left(\mathcal{G} - 8G^{\mu\nu}\phi_\mu\phi_\nu - 12\Box\phi(\nabla\phi)^2 - 12(\nabla\phi)^4\right)\Big\},$$

the metric function and scalar field are given by

$$f(r) = \frac{r^2}{2\alpha_4}\left(1 \pm \sqrt{1 + 4\alpha_4\left(\frac{\Lambda}{3} + \frac{2M}{r^3}\right)}\right)$$

$$\phi(r) = \ln\left(\frac{\eta}{r}\right)$$

where M and η are constants of integration.

Curiosities :

1. The solution does not depend on the coupling α_5 since the $T_{\mu\nu}^{(\alpha_5)}$ associated to α_5 vanishes on-shell (it is a stealth only on this part).
2. The integration constant η does not appear in the metric solution, and this can be explained by the fact that the α_4-part is shift symmetric in ϕ i. e. $\phi \to \phi + \text{cst}$

Other Solution

Considering $\beta_4 = \lambda_4 = \alpha_4 = 0$, and through a redefinition of the scalar field $\Phi^{\frac{2}{3}} = e^{\phi}$, the theory is given by

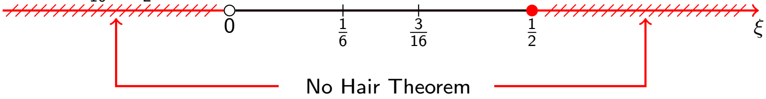
$$S = \int d^4x\sqrt{-g}\Big\{R - 2\Lambda - 2\lambda_5\Phi^{\frac{10}{3}} - \frac{1}{2}(\partial\Phi)^2 - \frac{3}{32}R\Phi^2 -$$

$$-\alpha_5\left(\mathcal{G} - \frac{32}{9}\frac{G^{\mu\nu}\Phi_\mu\Phi_\nu}{\Phi^{\frac{4}{3}}} - \frac{32}{9}\frac{\Box\Phi(\nabla\Phi)^2}{\Phi^{\frac{7}{3}}} - \frac{64}{27}\frac{(\nabla\Phi)^4}{\Phi^{\frac{10}{3}}}\right)\Big\}.$$

We obtain the following solution

$$f(r) = \frac{1}{1+\frac{4\alpha_5\eta\kappa}{3r^3}}\left[\kappa - \frac{\Lambda r^2}{3} - \frac{2M}{r} - \frac{4\alpha_5\eta\kappa^2}{15r^3}\right], \quad \Phi(r) = \frac{\eta}{r}.$$

Here $\xi = \frac{3}{16} < \frac{1}{2}$



From static to Vaidya-like solutions

Considering the metric function as a function that depends also on the retarded (advanced) time u,

$$ds^2 = -f(u,r)du^2 - 2dudr + \frac{r^2 d\theta^2}{1-\kappa\theta^2} + r^2\theta^2 dy^2,$$
$$\phi(r) = \ln\left(\frac{\eta}{r}\right),$$

where η is fixed by the compatibility conditions and the equation E_{rr} is automatically satisfied.

Evaluating the scalar field $\phi(r)$ and the compatibility conditions in the Einstein equations E_{uu} one gets,

$$E_{uu} = \partial_r \Bigg(\frac{\alpha_4 f^2}{r} - \left(r + \frac{4\alpha_5 \eta\kappa}{3r^2} + \frac{2\alpha_4\kappa}{r} \right) f + \frac{4\eta\alpha_5\kappa^2}{15 r^2} - \frac{\alpha_4\kappa^2}{} + \frac{1}{r}\left(\lambda_4\eta^4 + \beta_4\,\kappa\,\eta^2\right) - \frac{r^3\Lambda}{3} + \kappa r \Bigg) - \frac{1}{f}\partial_u \left(\frac{\alpha_4 f^2}{r} - \left(r + \frac{4\alpha_5\eta\kappa}{3r^2} + \frac{2\alpha_4\kappa}{r}\right) f \right).$$

$$= (\partial_r - \frac{1}{f(u,r)}\partial_u) E^{\text{static}}(f(u,r)),$$

$$E_{\theta\theta} = \partial_{rr} E^{\text{static}}(f(u,r)),$$

$$\Longrightarrow E^{\text{static}}(f(u,r)) = C_1(u) r + 2M(u),$$

$$\Longrightarrow E_{uu} = \frac{C_1(u)}{r^2} - \frac{\dot{C}_1(u)}{r\,f(u,r)} - \frac{2\dot{M}(u)}{r^2\,f(u,r)},$$

it is easy to see that choosing $C_1(u) = 0$ leads to the generalized Vaidya relation

$$E_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu} = -\frac{2\dot{M}(u)}{r^2}\delta^u_\mu \delta^u_\nu,$$

Hence, we have shown that the compatibility conditions can be generalized to accommodate a time dependence by promoting the constant mass to a function of the retarded (advanced) time $\Longrightarrow$ As a consequence, the static black hole solutions can be naturally promoted to Vaidya type solutions with metric

$$f(u,r)=\kappa+\frac{2\alpha_5\eta\kappa}{3r\alpha_4}+\frac{r^2}{2\alpha_4}\left(1\pm\sqrt{\left(1+\frac{4\alpha_5\eta\kappa}{3r^3}\right)^2+4\alpha_4\left(\frac{\Lambda}{3}+\frac{2M(u)}{r^3}+\frac{2\alpha_4\kappa^2}{r^4}+\frac{8\alpha_5\eta\kappa^2}{5r^5}\right)}\right)$$

Let us now define

$$\mathcal{L}_1(n)=2e^{n\phi},\qquad \mathcal{L}_2(n)=e^{(n-2),\phi}\left[R+(n-1)(n-2)(\nabla\phi)^2\right]$$

$$\mathcal{L}_3(n)=e^{(n-4)\phi}\left[\mathcal{G}-4(n-3)(n-4)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-2(n-2)(n-3)(n-4)\Box\phi(\nabla\phi)^2\right.$$
$$\left.-(n-2)(n-3)^2(n-4)(\nabla\phi)^4\right]$$

where $\mathcal{L}_1(n)$ (resp. $\mathcal{L}_2(n)$ and $\mathcal{L}_3(n)$) is conformally invariant in any dimension $n\geq 2$ (resp. $n\geq 3$ and $n\geq 4$).

General case with μ constant and $\mu \neq 1$

The resulting action for μ constant and $\mu \neq 1$

$$S_{\mu\neq 1} = \frac{1}{2}\int d^4x\sqrt{-g}\left\{\alpha\left[2(\mu-2)c^{2(\mu-3)}\mathcal{L}_1(6-2\mu)+4c^{2(\mu-2)}\mathcal{L}_2(6-2\mu)+\frac{c^{2(\mu-1)}}{(\mu-1)}\mathcal{L}_3(6-2\mu)\right]\right.$$
$$-\gamma\left[\frac{12}{(2\mu+3)c^5}\mathcal{L}_1(5)-\frac{4}{c^3}\mathcal{L}_2(5)+\frac{(2\mu+1)}{c}\mathcal{L}_3(5)\right]$$
$$\left.+2(\mu-1)c^{2(\mu-2)}\mathcal{L}_1(4-2\mu)+2c^{2(\mu-1)}\mathcal{L}_2(4-2\mu)\right\}.$$

The metric solution is given by

$$f(u,r) = \frac{1}{2\mu-1}\left[1+\frac{\gamma r^{1-2\mu}}{\alpha}+\frac{r^2}{2\alpha}\left(1\pm\sqrt{H(r)}\right)\right]$$

with

$$H(r) = \left(1+\frac{2\gamma}{r^{2\mu+1}}\right)^2+\frac{4\alpha\Lambda}{3}+\frac{8\alpha M}{r^{2\mu+1}}+\frac{16\alpha\gamma(2\mu+1)}{(2\mu+3)r^{2\mu+3}}-\frac{8\alpha^2}{(2\mu-3)r^4}$$

As in the $\mu = 1$ case, its extension to generalized Vaidya-like solution $M \to M(u)$ yields

$$E_{\mu\nu} := G_{\mu\nu}+\Lambda g_{\mu\nu}-T_{\mu\nu} = -\frac{2\,\mu\,\dot{M}(u)}{(2\,\mu-1)\,r^2}\delta^u_\mu\delta^u_\nu,$$

Conclusions :

1. Quite general Horndeski theories with arbitrary $\phi-$dependent potentials and without any apparent symmetries which admit interesting and explicit black hole solutions.
2. The resulting action turns out to have a conformal origin (non-Noetherian in four dimension and Noetherian in five dimensions). Understand this origin.
3. Linear and quadratic black hole solutions which can be promoted to generalized Vaidya-like configurations $M \to M(u)$.
4. We have completely specified the solutions for a constant factor of proportionality. What is for $\mu \neq$ cst (for example the BBMB solution corresponds to $\mu \neq$ cst).